

# Guiding-center polarization and magnetization effects in gyrokinetic theory

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Higher-order guiding-center polarization and magnetization effects are introduced in gyrokinetic theory by keeping first-order terms in background magnetic-field nonuniformity. These results confirm the consistency of the two-step perturbation analysis used in modern gyrokinetic theory.

## I. INTRODUCTION

Polarization and magnetization effects play a fundamental role in modern gyrokinetic theory [1, 2]. The standard form of modern gyrokinetic theory is derived by a two-step Lie-transform perturbation analysis that retains the effects of first-order ( $\epsilon_B$ ) guiding-center drifts associated with the nonuniformity of the background magnetic field and gyrocenter effects up to second-order ( $\epsilon_\delta^2$ ) in electromagnetic-field fluctuations that perturb the background guiding-center plasma. First-order ( $\epsilon_\delta$ ) gyrocenter polarization and magnetization effects (which result from terms of order  $\epsilon_\delta^2$  in the gyrocenter Hamiltonian) are fully retained in nonlinear gyrokinetic theory. Because the ordering parameters  $\epsilon_B$  and  $\epsilon_\delta$  are often comparable ( $\epsilon_B \sim \epsilon_\delta$ ) in many practical applications of gyrokinetic theory, however, it is sometimes argued [3] that first-order ( $\epsilon_B$ ) guiding-center polarization and magnetization effects (which result from terms of order  $\epsilon_B \epsilon_\delta$  in the gyrocenter Hamiltonian) should also be retained in nonlinear gyrokinetic theory for a consistent treatment of polarization and magnetization effects.

The two-step derivation of modern gyrokinetic theory is based on a sequence of two near-identity phase-space transformations  $\mathbf{z}_0 \rightarrow \mathbf{z} \rightarrow \bar{\mathbf{z}}$  from local particle coordinates  $\mathbf{z}_0 \equiv (\mathbf{x}, v_{0\parallel}, \mu_0, \theta_0)$  to guiding-center coordinates  $\mathbf{z} \equiv (\mathbf{X}, v_{\parallel}, \mu, \theta)$  and then to gyrocenter coordinates  $\bar{\mathbf{z}} \equiv (\bar{\mathbf{X}}, \bar{v}_{\parallel}, \bar{\mu}, \bar{\theta})$ . The purpose of the guiding-center transformation [4, 5]  $\mathbf{z}_0 \rightarrow \mathbf{z}$  (with small parameter  $\epsilon_B$ ) is to asymptotically decouple the fast gyromotion of charged particles in a strong weakly-nonuniform ( $\epsilon_B \ll 1$ ) background magnetic field and construct the guiding-center magnetic moment  $\mu \equiv \mu_0 + \epsilon_B \mu_1 + \dots$  as an adiabatic invariant (from the local particle magnetic moment  $\mu_0 \equiv m|\mathbf{v}_{0\perp}|^2/2B$ ). The introduction of low-frequency electromagnetic-field fluctuations destroy the adiabatic invariance of the guiding-center magnetic moment  $\mu$ , which requires the gyrocenter transformation  $\mathbf{z} \rightarrow \bar{\mathbf{z}}$  (with small parameter  $\epsilon_\delta$ ) in order to restore the adiabatic invariance of the gyrocenter magnetic moment  $\bar{\mu} \equiv \mu + \epsilon_\delta \bar{\mu}_1 + \dots$ .

Each transformation introduces a polarization charge density in the gyrokinetic Poisson equation and polarization and magnetization current densities in the gyrokinetic Ampère equation [6]. These effects explicitly involve the generalized gyroradius  $\bar{\boldsymbol{\rho}} \equiv \mathbf{x} - \bar{\mathbf{X}}$  defined as the displacement of the gyrocenter position  $\bar{\mathbf{X}}$  from the particle position  $\mathbf{x}$ . As a result of the guiding-center and gyrocenter transformations, the generalized gyroradius

$$\bar{\boldsymbol{\rho}} \equiv \left( \boldsymbol{\rho}_{0\text{gc}} + \epsilon_B \boldsymbol{\rho}_{1\text{gc}} + \dots \right) + \left( \epsilon_\delta \bar{\boldsymbol{\rho}}_{1\text{gy}} + \dots \right) \quad (1)$$

is decomposed into the guiding-center gyroradius  $\boldsymbol{\rho}_{\text{gc}} \equiv \boldsymbol{\rho}_{0\text{gc}} + \epsilon_B \boldsymbol{\rho}_{1\text{gc}} + \dots$  and the gyrocenter gyroradius  $\bar{\boldsymbol{\rho}}_{\text{gy}} \equiv \epsilon_\delta \bar{\boldsymbol{\rho}}_{1\text{gy}} + \dots$  (which vanishes in the absence of fluctuations). As will be discussed below, polarization effects are associated with the gyrorangle-averaged displacement  $\langle \bar{\boldsymbol{\rho}} \rangle$  while magnetization effects are associated with the gyrorangle-dependent displacement  $\tilde{\boldsymbol{\rho}} \equiv \bar{\boldsymbol{\rho}} - \langle \bar{\boldsymbol{\rho}} \rangle$ .

The first-order ( $\epsilon_\delta$ ) gyrocenter polarization and magnetization effects associated with  $\bar{\boldsymbol{\rho}}_{1\text{gy}}$  (which can also include terms of arbitrary order in  $\epsilon_B$ ) have been discussed elsewhere [1, 2]. Moreover, we note that, since electromagnetic-field fluctuations satisfy the gyrokinetic ordering  $|\boldsymbol{\rho}_{0\text{gc}} \cdot \nabla| \sim 1$  (i.e., perpendicular wavelengths are of the same order as the lowest-order gyroradius), the guiding-center gyroradius-expansions considered in the present work apply only to the derivations of guiding-center polarization and magnetization effects. Hence, the first-order ( $\epsilon_\delta$ ) gyrocenter polarization and magnetization effects will be ignored in what follows.

Because the lowest-order guiding-center gyroradius is explicitly gyroangle-dependent (i.e.,  $\langle \boldsymbol{\rho}_{0\text{gc}} \rangle \equiv 0$ ), there is no zeroth-order term in the guiding-center polarization while the guiding-center magnetization has a non-vanishing zeroth-order term. First-order guiding-center polarization effects were investigated previously [7–10] outside the context of gyrokinetic theory. The purpose of the present paper is to investigate the higher-order polarization and magnetization effects introduced by the guiding-center gyroradius  $\boldsymbol{\rho}_{\text{gc}} = \boldsymbol{\rho}_{0\text{gc}} + \epsilon_B \boldsymbol{\rho}_{1\text{gc}} + \dots$  within the standard format of nonlinear gyrokinetic theory.

The remainder of the present paper is organized as follows. In Sec. II, we introduce variational definitions of the reduced polarization and magnetization used in modern gyrokinetic theory based on functional derivatives of the

gyrocenter Hamiltonian. In Sec. III, we explicitly make use of the higher-order correction  $\boldsymbol{\rho}_{1gc}$  to the guiding-center gyroradius  $\boldsymbol{\rho}_{gc}$  to derive explicit expressions for the first-order guiding-center polarization and magnetization. These results confirm that the two-step perturbation analysis used in modern gyrokinetic theory [1] yield a consistent set of gyrokinetic Vlasov-Maxwell equations that include all first-order ( $\epsilon_B$  and  $\epsilon_\delta$ ) polarization and magnetization effects.

## II. GYROCENTER POLARIZATION AND MAGNETIZATION

The polarization effects in gyrokinetic theory are formally defined in terms of the relation (at a fixed position  $\mathbf{r}$  at time  $t$ ) between the particle charge density  $\varrho(\mathbf{r}, t)$  and the gyrocenter charge density  $\varrho_{gy}(\mathbf{r}, t)$ :

$$\varrho \equiv \varrho_{gy} - \nabla \cdot \mathbf{P}_{gy}, \quad (2)$$

where the polarization charge density  $\varrho_{pol} \equiv -\nabla \cdot \mathbf{P}_{gy}$  serves as a definition of the gyrocenter polarization  $\mathbf{P}_{gy}$ . The gyrocenter charge density (summation over particle species is implied)

$$\varrho_{gy} \equiv e \int \overline{F} d^3\overline{v} \quad (3)$$

is defined as a gyrocenter-velocity-space integral of the gyroangle-independent gyrocenter Vlasov distribution  $\overline{F}$  (where  $d^3\overline{v} \equiv 2\pi \overline{\mathcal{J}} d\overline{v}_\parallel d\overline{\mu}$  with Jacobian  $\overline{\mathcal{J}}$  to be defined below and the gyrocenter gyroangle integration has been performed explicitly). The gyrocenter polarization

$$\mathbf{P}_{gy} \equiv \mathbf{P}_{gc} + \epsilon_\delta \mathbf{P}_{1gy} + \dots, \quad (4)$$

on the other hand, is defined in terms of the guiding-center polarization

$$\mathbf{P}_{gc} \equiv \epsilon_B \mathbf{P}_{1gc} + \dots, \quad (5)$$

which vanishes in a uniform magnetized plasma, and the first-order gyrocenter polarization  $\epsilon_\delta \mathbf{P}_{1gy}$ , which vanishes in the absence of field fluctuations.

The magnetization effects, on the other hand, are formally defined in terms of the relation (at a fixed position  $\mathbf{r}$  at time  $t$ ) between the particle current density  $\mathbf{J}(\mathbf{r}, t)$  and the gyrocenter current density  $\mathbf{J}_{gy}(\mathbf{r}, t)$ :

$$\mathbf{J} \equiv \mathbf{J}_{gy} + \frac{\partial \mathbf{P}_{gy}}{\partial t} + c \nabla \times \mathbf{M}_{gy}, \quad (6)$$

where the gyrocenter polarization current  $\mathbf{J}_{pol} \equiv \partial \mathbf{P}_{gy} / \partial t$  is defined in terms of the gyrocenter polarization (4) and the magnetization current  $\mathbf{J}_{mag} \equiv c \nabla \times \mathbf{M}_{gy}$  serves as a definition of the gyrocenter magnetization  $\mathbf{M}_{gy}$ . The gyrocenter current density

$$\mathbf{J}_{gy} \equiv \mathbf{J}_{gc} + \epsilon_\delta \mathbf{J}_{1gy} + \dots \quad (7)$$

is defined in terms of the guiding-center current density

$$\mathbf{J}_{gc} \equiv e \int \overline{F} \frac{d_{gc} \overline{\mathbf{X}}}{dt} d^3\overline{v} = \mathbf{J}_{0gc} + \epsilon_B \mathbf{J}_{1gc} + \dots, \quad (8)$$

which includes first-order guiding-center drifts, while the gyrocenter magnetization

$$\mathbf{M}_{gy} \equiv \mathbf{M}_{gc} + \epsilon_\delta \mathbf{M}_{1gy} + \dots \quad (9)$$

is defined in terms of the guiding-center magnetization

$$\mathbf{M}_{gc} \equiv \mathbf{M}_{0gc} + \epsilon_B \mathbf{M}_{1gc} + \dots, \quad (10)$$

where the non-vanishing zeroth-order guiding-center magnetization

$$\mathbf{M}_{0gc} \equiv - \left( \int \overline{\mu} \overline{F} d^3\overline{v} \right) \hat{\mathbf{b}} \quad (11)$$

is a well-known result [9, 11].

Note that the definitions (2) and (6) ensure that the gyrokinetic Vlasov-Maxwell equations satisfy the gyrocenter charge conservation law  $\partial \varrho_{\text{gy}} / \partial t + \nabla \cdot \mathbf{J}_{\text{gy}} = 0$ , since polarization effects  $\partial \varrho_{\text{pol}} / \partial t + \nabla \cdot \mathbf{J}_{\text{pol}} \equiv 0$  conserve charge identically while the magnetization current density is divergenceless  $\nabla \cdot \mathbf{J}_{\text{mag}} \equiv 0$ . We also note that the quasineutrality condition  $\varrho \equiv 0$  used in gyrokinetic theory [1] can now be expressed (up to order  $\epsilon_\delta$  and  $\epsilon_B$ ) as

$$\varrho_{\text{gy}} - \nabla \cdot \mathbf{P}_{\text{gc}} = \epsilon_\delta \nabla \cdot \mathbf{P}_{1\text{gy}}, \quad (12)$$

where the left side contains the standard polarization effects associated with the lowest-order guiding-center gyroradius  $\boldsymbol{\rho}_{0\text{gc}}$  as well as the higher-order polarization effects associated with the first-order guiding-center gyroradius  $\boldsymbol{\rho}_{1\text{gc}}$  and gradients  $\nabla \boldsymbol{\rho}_{0\text{gc}}$ .

### A. Functional Definitions

The gyrocenter polarization (4) and magnetization (9) can be defined as functionals of the gyrocenter Vlasov distribution  $\bar{F}$  and functional derivatives of the gyrocenter Hamiltonian as follows [1, 12]. Here, the gyrocenter Hamiltonian

$$H_{\text{gy}} \equiv H_{\text{gc}} + \epsilon_\delta H_{1\text{gy}} + \epsilon_\delta^2 H_{2\text{gy}} + \dots \quad (13)$$

is formally expressed as an asymptotic expansion in powers of  $\epsilon_B$  and  $\epsilon_\delta$ . The unperturbed gyrocenter Hamiltonian is defined as the guiding-center Hamiltonian [4]

$$H_{\text{gc}} \equiv \frac{m}{2} \bar{v}_\parallel^2 + \bar{\mu} B, \quad (14)$$

where higher-order corrections ( $\epsilon_B^n$ , for  $n \geq 1$ ) can be made to vanish [5]. The first-order gyrocenter Hamiltonian (in the Hamiltonian representation of gyrokinetic theory in which the guiding-center Poisson bracket is left unperturbed) is defined as [13]

$$H_{1\text{gy}} \equiv \left\langle e \Phi_1(\bar{\mathbf{X}} + \boldsymbol{\rho}_{\text{gc}}, t) - \frac{e}{c} \mathbf{A}_1(\bar{\mathbf{X}} + \boldsymbol{\rho}_{\text{gc}}, t) \cdot \left( \frac{d_{\text{gc}} \bar{\mathbf{X}}}{dt} + \frac{d_{\text{gc}} \boldsymbol{\rho}_{\text{gc}}}{dt} \right) \right\rangle, \quad (15)$$

where the first-order electromagnetic potentials ( $\Phi_1, \mathbf{A}_1$ ) represent the electromagnetic-field fluctuations that perturb the guiding-center plasma, and the guiding-center (unperturbed) evolution operator  $d_{\text{gc}}/dt \equiv \partial/\partial t + \{, H_{\text{gc}}\}_{\text{gc}}$  is defined in terms of the guiding-center Hamiltonian (14) and the guiding-center Poisson bracket  $\{, \}_{\text{gc}}$  (to be defined below). The exact expression for the second-order gyrocenter Hamiltonian  $H_{2\text{gy}}$  has been given elsewhere [1, 13] and will not be needed in what follows since we will not be concerned with first-order ( $\epsilon_\delta$ ) gyrocenter polarization and magnetization.

The first-order gyrocenter Hamiltonian (15), which can be expanded as  $H_{1\text{gy}} \equiv H_{1\text{gy}}^{(0)} + \epsilon_B H_{1\text{gy}}^{(1)} + \dots$ , contains the lowest-order terms [13]

$$H_{1\text{gy}}^{(0)} \equiv \left\langle e \Phi_{1\text{gc}} - \frac{e}{c} \mathbf{A}_{1\text{gc}} \cdot \left( \bar{v}_\parallel \hat{\mathbf{b}} + \Omega \frac{\partial \boldsymbol{\rho}_{0\text{gc}}}{\partial \bar{\theta}} \right) \right\rangle, \quad (16)$$

where  $\Phi_{1\text{gc}} \equiv \Phi_1(\bar{\mathbf{X}} + \boldsymbol{\rho}_{0\text{gc}}, t)$  and  $\mathbf{A}_{1\text{gc}} \equiv \mathbf{A}_1(\bar{\mathbf{X}} + \boldsymbol{\rho}_{0\text{gc}}, t)$ , as well as the first-order guiding-center corrections

$$\begin{aligned} H_{1\text{gy}}^{(1)} \equiv & - \left\langle \boldsymbol{\rho}_{1\text{gc}} \cdot \left[ e \mathbf{E}_{1\text{gc}} + \frac{e}{c} \left( \bar{v}_\parallel \hat{\mathbf{b}} \times \mathbf{B}_{1\perp\text{gc}} + \Omega \frac{\partial \boldsymbol{\rho}_{0\text{gc}}}{\partial \bar{\theta}} \times B_{1\parallel\text{gc}} \hat{\mathbf{b}} \right) \right] \right\rangle \\ & - \frac{e}{c} \left\langle \mathbf{A}_{1\text{gc}} \cdot \left( \frac{d_{\text{gc}}^{(1)} \bar{\mathbf{X}}}{dt} + \frac{d_{\text{gc}}^{(1)} \boldsymbol{\rho}_{0\text{gc}}}{dt} \right) \right\rangle \end{aligned} \quad (17)$$

where  $\mathbf{E}_{1\text{gc}} \equiv -\bar{\nabla} \Phi_{1\text{gc}} - c^{-1} \partial \mathbf{A}_{1\text{gc}} / \partial t$  and  $\mathbf{B}_{1\text{gc}} \equiv \nabla \times \mathbf{A}_{1\text{gc}} = B_{1\parallel\text{gc}} \hat{\mathbf{b}} + \mathbf{B}_{1\perp\text{gc}}$ , while  $d_{\text{gc}}^{(1)} \bar{\mathbf{X}}/dt$  and  $d_{\text{gc}}^{(1)} \boldsymbol{\rho}_{0\text{gc}}/dt$  denote first-order ( $\epsilon_B$ ) corrections to  $\bar{v}_\parallel \hat{\mathbf{b}}$  and  $\Omega \partial \boldsymbol{\rho}_{0\text{gc}} / \partial \bar{\theta}$ , respectively.

### 1. Gyrocenter polarization

The gyrocenter charge density and the gyrocenter polarization are defined as [12]

$$\rho_{\text{gy}} - \nabla \cdot \mathbf{P}_{\text{gy}} \equiv \epsilon_{\delta}^{-1} \int \bar{F} \frac{\delta H_{\text{gy}}}{\delta \Phi_1(\mathbf{r}, t)} d^6 \bar{\mathbf{z}} = \int \bar{F} \frac{\delta H_{1\text{gy}}}{\delta \Phi_1(\mathbf{r}, t)} d^6 \bar{\mathbf{z}} + \dots, \quad (18)$$

where guiding-center polarization effects (5) are defined by the identity

$$- \nabla \cdot \mathbf{P}_{\text{gc}} \equiv \int \bar{F} \left[ e \langle \delta^3(\bar{\mathbf{X}} + \boldsymbol{\rho}_{\text{gc}} - \mathbf{r}) \rangle - e \delta^3(\bar{\mathbf{X}} - \mathbf{r}) \right] d^6 \bar{\mathbf{z}}. \quad (19)$$

By expanding Eq. (19) in powers of  $\boldsymbol{\rho}_{\text{gc}}$  and integrating by parts to eliminate  $\delta^3(\bar{\mathbf{X}} - \mathbf{r})$ , we obtain the guiding-center polarization

$$\mathbf{P}_{\text{gc}} \equiv e \int \bar{F} \langle \boldsymbol{\rho}_{\text{gc}} \rangle d^3 \bar{\mathbf{v}} - \nabla \cdot \left( \frac{e}{2} \int \bar{F} \langle \boldsymbol{\rho}_{\text{gc}} \boldsymbol{\rho}_{\text{gc}} \rangle d^3 \bar{\mathbf{v}} \right) + \dots, \quad (20)$$

where dipole ( $\langle \boldsymbol{\rho}_{\text{gc}} \rangle$ ) and quadrupole ( $\langle \boldsymbol{\rho}_{\text{gc}} \boldsymbol{\rho}_{\text{gc}} \rangle$ ) contributions are shown. By using the fact that  $\langle \boldsymbol{\rho}_{0\text{gc}} \rangle \equiv 0$ , i.e., Eq. (20) has a vanishing zeroth-order term, the first-order guiding-center polarization is defined in terms of the functionals

$$\mathbf{P}_{1\text{gc}} \equiv e \int \left\{ \bar{F} \left[ \langle \boldsymbol{\rho}_{1\text{gc}} \rangle - \nabla \cdot \left( \left\langle \frac{\boldsymbol{\rho}_{0\text{gc}} \boldsymbol{\rho}_{0\text{gc}}}{2} \right\rangle \right) \right] - \left\langle \frac{\boldsymbol{\rho}_{0\text{gc}} \boldsymbol{\rho}_{0\text{gc}}}{2} \right\rangle \cdot \nabla \bar{F} \right\} d^3 \bar{\mathbf{v}}, \quad (21)$$

where we have grouped terms that directly involve the background magnetic-field nonuniformity and the term that directly involves the spatial gradient of the gyrocenter Vlasov distribution  $\bar{F}$ . The latter term (whose ordering is assumed to be comparable to  $\epsilon_B$ ) is generated by the Taylor expansion of the integral

$$\int \bar{F} \langle \delta^3(\bar{\mathbf{X}} + \boldsymbol{\rho}_{0\text{gc}} - \mathbf{r}) \rangle d^6 \bar{\mathbf{z}} = \int \left( \bar{F} + \frac{\bar{\mu} B}{2 m \Omega^2} \nabla_{\perp}^2 \bar{F} + \dots \right) d^3 \bar{\mathbf{v}}, \quad (22)$$

where  $\boldsymbol{\rho}_{0\text{gc}}$  is assumed to be spatially uniform. We will return to Eq. (21) once we have obtained an expression for the first-order guiding-center gyroradius  $\boldsymbol{\rho}_{1\text{gc}}$  in the next Section.

### 2. Gyrocenter magnetization

The gyrocenter current density and the gyrocenter magnetization are defined as

$$\mathbf{J}_{\text{gy}} + \frac{\partial \mathbf{P}_{\text{gy}}}{\partial t} + c \nabla \times \mathbf{M}_{\text{gy}} \equiv \epsilon_{\delta}^{-1} \int \bar{F} \left( -c \frac{\delta H_{\text{gy}}}{\delta \mathbf{A}_1(\mathbf{r}, t)} \right) d^6 \bar{\mathbf{z}} = \int \bar{F} \left( -c \frac{\delta H_{1\text{gy}}}{\delta \mathbf{A}_1(\mathbf{r}, t)} \right) d^6 \bar{\mathbf{z}} + \dots, \quad (23)$$

where guiding-center magnetization effects (10) are defined by the identity

$$\mathbf{J}_{\text{gc}} + \frac{\partial \mathbf{P}_{\text{gc}}}{\partial t} + c \nabla \times \mathbf{M}_{\text{gc}} \equiv \int \bar{F} \left[ e \frac{d_{\text{gc}} \bar{\mathbf{X}}}{dt} \left\langle \delta^3(\bar{\mathbf{X}} + \boldsymbol{\rho}_{\text{gc}} - \mathbf{r}) \right\rangle + e \left\langle \frac{d_{\text{gc}} \boldsymbol{\rho}_{\text{gc}}}{dt} \delta^3(\bar{\mathbf{X}} + \boldsymbol{\rho}_{\text{gc}} - \mathbf{r}) \right\rangle \right] d^6 \bar{\mathbf{z}}, \quad (24)$$

with the guiding-center current density  $\mathbf{J}_{\text{gc}}$  defined by Eq. (8). After expanding Eq. (24) in powers of  $\boldsymbol{\rho}_{\text{gc}}$  and carrying out several manipulations (see Appendix A for details), we obtain the guiding-center magnetization

$$\mathbf{M}_{\text{gc}} \equiv \frac{e}{c} \int \bar{F} \left[ \frac{1}{2} \left\langle \boldsymbol{\rho}_{\text{gc}} \times \frac{d_{\text{gc}} \boldsymbol{\rho}_{\text{gc}}}{dt} \right\rangle + \langle \boldsymbol{\rho}_{\text{gc}} \rangle \times \frac{d_{\text{gc}} \bar{\mathbf{X}}}{dt} \right] d^3 \bar{\mathbf{v}}, \quad (25)$$

which is naturally split into an intrinsic contribution associated with  $(1/2) \langle \boldsymbol{\rho}_{\text{gc}} \times d_{\text{gc}} \boldsymbol{\rho}_{\text{gc}} / dt \rangle$  and a moving-electric-dipole contribution  $\langle \boldsymbol{\rho}_{\text{gc}} \rangle \times d_{\text{gc}} \bar{\mathbf{X}} / dt$ . Note that the moving-magnetic-dipole contribution to the guiding-center polarization (20) is a relativistic effect [14] which falls outside the scope of the present work.

The guiding-center magnetization has a well-known zeroth-order intrinsic contribution [11]

$$\mathbf{M}_{0\text{gc}} \equiv \frac{e}{2c} \int \bar{F} \left\langle \boldsymbol{\rho}_{0\text{gc}} \times \left( \Omega \frac{\partial \boldsymbol{\rho}_{0\text{gc}}}{\partial \theta} \right) \right\rangle d^3 \bar{\mathbf{v}}, \quad (26)$$

from which we recover the classical result (11), where we used the lowest-order identities  $\partial \rho_{0gc}/\partial \bar{\theta} \equiv \rho_{0gc} \times \hat{\mathbf{b}}$  and  $|\rho_{0gc}|^2 \equiv 2\bar{\mu}B/(m\Omega^2)$ . The first-order contribution

$$\begin{aligned} \mathbf{M}_{1gc} = & \frac{e}{c} \int \bar{F} \left[ \left\langle \tilde{\rho}_{1gc} \times \left( \Omega \frac{\partial \rho_{0gc}}{\partial \bar{\theta}} \right) \right\rangle + \frac{1}{2} \left\langle \rho_{0gc} \times \left( \frac{d_{gc}^{(1)} \rho_{0gc}}{dt} \right) \right\rangle \right] d^3 \bar{v} \\ & + \frac{e}{c} \int \bar{F} \left[ \langle \rho_{1gc} \rangle \times (\bar{v}_{\parallel} \hat{\mathbf{b}}) \right] d^3 \bar{v}, \end{aligned} \quad (27)$$

on the other hand, is decomposed in terms of the first-order corrections (the first two terms) to the zeroth-order guiding-center intrinsic magnetization (26), while the third term yields the moving-electric-dipole contribution since it involves the gyroangle-averaged first-order guiding-center gyroradius  $\langle \rho_{1gc} \rangle$  appearing in Eq. (21).

### III. GUIDING-CENTER POLARIZATION AND MAGNETIZATION

In the present Section, we make use of the higher-order corrections to the guiding-center transformation [4, 5] for the purpose of determining the first-order guiding-center gyroradius  $\rho_{1gc}$  used in the first-order guiding-center polarization (21) and the first-order guiding-center magnetization (27). We use Northrop's macroscopic interpretation [15] of the small parameter  $\epsilon_B$  which, for finite macroscopic length  $L_B$ , allows us to use  $e^{-1} \sim \epsilon$  as an ordering parameter. Hence, the guiding-center dynamical reduction is generated by the near-identity phase-space transformation

$$z^\alpha \equiv z_0^\alpha + \epsilon G_1^\alpha + \epsilon^2 \left( G_2^\alpha + \frac{1}{2} G_1^\beta \frac{\partial G_1^\alpha}{\partial z^\beta} \right) + \dots, \quad (28)$$

where  $\epsilon \sim e^{-1}$  is used as an ordering parameter and the phase-space vector field  $\mathbf{G}_n$  is said to generate the phase-space transformation at order  $n \geq 1$ .

#### A. Guiding-center phase-space transformation

The guiding-center coordinates  $z^\alpha = (\mathbf{X}, v_{\parallel}, \mu, \theta)$  are defined (up to first order in  $\epsilon_B$ ) as [5]

$$\mathbf{X} = \mathbf{x} - \epsilon \rho_0 + \epsilon^2 \left[ G_2^\mathbf{x} + \frac{1}{2} \rho_0 \cdot \nabla \rho_0 - \frac{1}{2} \left( G_1^\mu \frac{\partial \rho_0}{\partial \mu_0} + G_1^\theta \frac{\partial \rho_0}{\partial \theta_0} \right) \right], \quad (29)$$

$$v_{\parallel} = v_{0\parallel} \left[ 1 - \epsilon \rho_0 \cdot (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \right] + \epsilon \frac{\mu_0 B}{m\Omega} \left( \tau + \mathbf{a}_1 : \nabla \hat{\mathbf{b}} \right), \quad (30)$$

$$\mu = \mu_0 \left[ 1 - \epsilon \rho_{0\parallel} \left( \tau + \mathbf{a}_1 : \nabla \hat{\mathbf{b}} \right) \right] + \epsilon \rho_0 \cdot \left( \mu_0 \nabla \ln B + \frac{m v_{0\parallel}^2}{B} \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \right), \quad (31)$$

$$\theta = \theta_0 - \epsilon \rho_0 \cdot \mathbf{R} + \epsilon \rho_{0\parallel} \left( \mathbf{a}_2 : \nabla \hat{\mathbf{b}} \right) + \epsilon \frac{\partial \rho_0}{\partial \theta_0} \cdot \left( \nabla \ln B + \frac{m v_{0\parallel}^2}{2 \mu_0 B} \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \right), \quad (32)$$

where  $\rho_{0\parallel} \equiv v_{0\parallel}/(m\Omega)$  denotes the “parallel” gyroradius,  $\tau \equiv \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}$  denotes the background magnetic torsion,  $\mathbf{R}$  denotes the gyrogauging vector field, and expressions for the gyroangle-dependent dyadic tensors  $(\mathbf{a}_1, \mathbf{a}_2)$  are not needed in what follows. Here, the second-order spatial component  $G_2^\mathbf{x}$  is expressed as

$$G_2^\mathbf{x} = G_{2\parallel} \hat{\mathbf{b}} + \rho_{\parallel} \tau \rho_0 + \frac{1}{2} \left( G_1^\mu - \mu \rho_0 \cdot \nabla \ln B \right) \frac{\partial \rho_0}{\partial \mu_0} + \frac{1}{2} \left( G_1^\theta + \rho_0 \cdot \mathbf{R} \right) \frac{\partial \rho_0}{\partial \theta_0}. \quad (33)$$

where

$$G_{2\parallel} \equiv \hat{\mathbf{b}} \cdot G_2^\mathbf{x} = 2 \rho_{0\parallel} \frac{\partial \rho_0}{\partial \theta} \cdot (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) + \frac{\mu_0 B}{m\Omega^2} (\mathbf{a}_2 : \nabla \hat{\mathbf{b}}). \quad (34)$$

The first-order components  $(G_1^\mu, G_1^\theta)$ , defined in Eqs. (31)-(32) and used in Eqs. (29) and (33), are

$$G_1^\mu \equiv \rho_0 \cdot \left( \mu_0 \nabla \ln B + \frac{m v_{0\parallel}^2}{B} \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \right) - \mu_0 \rho_{0\parallel} (\tau + \mathbf{a}_1 : \nabla \hat{\mathbf{b}}), \quad (35)$$

$$G_1^\theta \equiv -\rho_0 \cdot \mathbf{R} + \rho_{0\parallel} (\mathbf{a}_2 : \nabla \hat{\mathbf{b}}) + \frac{\partial \rho_0}{\partial \theta_0} \cdot \left( \nabla \ln B + \frac{m v_{0\parallel}^2}{2 \mu_0 B} \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} \right). \quad (36)$$

The remaining first-order component defined in Eq. (30)

$$G_1^{v_{\parallel}} \equiv -v_{0\parallel} \boldsymbol{\rho}_0 \cdot (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) + \frac{\mu_0 B}{m\Omega} (\tau + \mathbf{a}_1 : \nabla \hat{\mathbf{b}}) \quad (37)$$

ensures that the first-order correction

$$H_{1gc} \equiv -G_1^{\alpha} \frac{\partial H_{gc}}{\partial z^{\alpha}} = \mu \boldsymbol{\rho}_0 \cdot \nabla B - mv_{\parallel} G_1^{v_{\parallel}} - B G_1^{\mu} \equiv 0 \quad (38)$$

to the guiding-center Hamiltonian (14) vanishes identically. The same construction algorithm can be applied at higher order (i.e.,  $H_{ngc} \equiv -G_n^{\alpha} \partial H_{gc} / \partial z^{\alpha} \equiv 0$  for  $n \geq 1$ ), so that the simplicity of the guiding-center Hamiltonian can be preserved in the form of Eq. (14) to all orders in  $\epsilon$  [4].

The Jacobian for the guiding-center phase-space transformation (29)-(32) is constructed from the Jacobian for the local particle phase-space coordinates  $\mathcal{J}_0 = B/m$  according to the formula

$$\begin{aligned} \mathcal{J} &\equiv \mathcal{J}_0 - \epsilon \frac{\partial}{\partial z^{\alpha}} (\mathcal{J}_0 G_1^{\alpha}) + \dots = \frac{B}{m} + \epsilon \left[ \nabla \cdot \left( \frac{B}{m} \boldsymbol{\rho}_0 \right) - \frac{B}{m} \left( \frac{\partial G_1^{v_{\parallel}}}{\partial v_{\parallel}} + \frac{\partial G_1^{\mu}}{\partial \mu} + \frac{\partial G_1^{\theta}}{\partial \theta} \right) \right] + \dots \\ &= \frac{B}{m} (1 + \epsilon \rho_{\parallel} \tau + \dots) \equiv \frac{B_{\parallel}^*}{m}. \end{aligned} \quad (39)$$

Hence, the near-identity guiding-center phase-space transformation is noncanonical since  $B_{\parallel}^* \neq B$ .

As a result of the guiding-center transformation (29)-(32), the guiding-center phase-space Lagrangian is expressed as

$$\Gamma_{gc} \equiv \left( \frac{e}{\epsilon c} \mathbf{A} + mv_{\parallel} \hat{\mathbf{b}} - \epsilon \mu \frac{B}{\Omega} \mathbf{R}^* \right) \cdot d\mathbf{X} + \epsilon \mu \frac{B}{\Omega} d\theta - H_{gc} dt, \quad (40)$$

where  $\mathbf{R}^* \equiv \mathbf{R} + (\tau/2) \hat{\mathbf{b}}$  and the guiding-center Hamiltonian  $H_{gc}$  is given by Eq. (14). The guiding-center Poisson bracket  $\{, \}_{gc}$  constructed from the symplectic part of the guiding-center phase-space Lagrangian (40) is [5]

$$\{F, G\}_{gc} = \epsilon^{-1} \frac{\Omega}{B} \left( \frac{\partial F}{\partial \theta} \frac{\partial G}{\partial \mu} - \frac{\partial F}{\partial \mu} \frac{\partial G}{\partial \theta} \right) + \frac{\mathbf{B}^*}{m B_{\parallel}^*} \cdot \left( \nabla^* F \frac{\partial G}{\partial v_{\parallel}} - \frac{\partial F}{\partial v_{\parallel}} \nabla^* G \right) - \epsilon \frac{\hat{\mathbf{c}} \mathbf{b}}{\epsilon B_{\parallel}^*} \cdot \nabla^* F \times \nabla^* G, \quad (41)$$

where  $\nabla^* \equiv \nabla + \mathbf{R}^* \partial / \partial \theta$ ,

$$\mathbf{B}^* \equiv \mathbf{B} + B \left( \epsilon \rho_{\parallel} \nabla \times \hat{\mathbf{b}} - \epsilon^2 \frac{\mu B}{m \Omega^2} \nabla \times \mathbf{R}^* + \dots \right), \quad (42)$$

and  $B_{\parallel}^* \equiv \hat{\mathbf{b}} \cdot \mathbf{B}^*$ . Note that, under the gyrogauging transformation  $\theta \rightarrow \theta' \equiv \theta + \psi(\mathbf{X})$ , the vector  $\mathbf{R}$  is gyrogauging-dependent (i.e.,  $\mathbf{R} \rightarrow \mathbf{R}' \equiv \mathbf{R} + \nabla \psi$ ), while the curl of  $\mathbf{R}$  is gyrogauging-invariant (i.e.,  $\nabla \times \mathbf{R}' = \nabla \times \mathbf{R}$ ). Hence, the guiding-center phase-space Lagrangian (40) and the guiding-center Poisson bracket (41) are both gyrogauging-invariant [4] since the combinations  $d\theta - \mathbf{R} \cdot d\mathbf{X}$  and  $\nabla + \mathbf{R} \partial / \partial \theta$  are gyrogauging-invariant.

Lastly, we can now write expressions for the velocities  $d_{gc} \mathbf{X} / dt$  and  $d_{gc} \boldsymbol{\rho}_0 / dt$  to be used in evaluating the first-order guiding-center magnetization (25). First, the guiding-center velocity

$$\frac{d_{gc} \mathbf{X}}{dt} \equiv \{\mathbf{X}, H_{gc}\}_{gc} = v_{\parallel} \hat{\mathbf{b}} + \frac{c \hat{\mathbf{b}}}{e B_{\parallel}^*} \times \left( \mu \nabla B + mv_{\parallel}^2 (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \right) \equiv v_{\parallel} \hat{\mathbf{b}} + \epsilon_B \frac{d_{gc}^{(1)} \mathbf{X}}{dt} \quad (43)$$

is expressed in terms of the zeroth-order motion along a magnetic-field line and first-order guiding-center drifts. Second, the gyration velocity

$$\frac{d_{gc} \boldsymbol{\rho}_0}{dt} \equiv \{\boldsymbol{\rho}_0, H_{gc}\}_{gc} = \Omega \frac{\partial \boldsymbol{\rho}_0}{\partial \theta} + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla^* \boldsymbol{\rho}_0 + \dots \equiv \Omega \frac{\partial \boldsymbol{\rho}_0}{\partial \theta} + \epsilon_B \frac{d_{gc}^{(1)} \boldsymbol{\rho}_0}{dt} + \dots \quad (44)$$

is expressed in terms of the zeroth-order perpendicular particle velocity and its first-order correction. The guiding-center velocities (43) and (44) appear in the first-order gyrocenter Hamiltonian (15) and, consequently, they contribute to the guiding-center magnetization (25).

### B. First-order corrections to the guiding-center gyroradius

So far we have not made a distinction between the gyroradius in particle phase space (labeled  $\boldsymbol{\rho}$ ) and the gyroradius in guiding-center phase space (labeled  $\boldsymbol{\rho}_{\text{gc}}$ ), which are respectively defined as

$$\left. \begin{aligned} \boldsymbol{\rho} &\equiv \mathbf{x} - \mathbb{T}_{\text{gc}} \mathbf{X} = \boldsymbol{\rho}_0 + \epsilon \boldsymbol{\rho}_1 + \cdots \\ \boldsymbol{\rho}_{\text{gc}} &\equiv \mathbb{T}_{\text{gc}}^{-1} \mathbf{x} - \mathbf{X} = \boldsymbol{\rho}_{0\text{gc}} + \epsilon \boldsymbol{\rho}_{1\text{gc}} + \cdots \end{aligned} \right\}. \quad (45)$$

In Eq. (45), the pull-back operator  $\mathbb{T}_{\text{gc}}$  transforms a function on guiding-center phase space into a function on particle phase space, while the push-forward operator  $\mathbb{T}_{\text{gc}}^{-1}$  transforms a function on particle phase space into a function on guiding-center phase space. From these definitions, we therefore obtain the relation between the particle and guiding-center gyroradii

$$\boldsymbol{\rho}_{\text{gc}} \equiv \mathbb{T}_{\text{gc}}^{-1} \boldsymbol{\rho} = \boldsymbol{\rho} - \epsilon G_1^\alpha \frac{\partial \boldsymbol{\rho}}{\partial z^\alpha} + \cdots \equiv \boldsymbol{\rho}_{0\text{gc}} + \epsilon \boldsymbol{\rho}_{1\text{gc}} + \cdots, \quad (46)$$

so that the lowest-order guiding-center gyroradius is  $\boldsymbol{\rho}_{0\text{gc}} \equiv \boldsymbol{\rho}_0$  is identical in both phase spaces. The first-order gyroradii  $\boldsymbol{\rho}_1$  and  $\boldsymbol{\rho}_{1\text{gc}} \equiv \boldsymbol{\rho}_1 - G_1^\alpha \partial \boldsymbol{\rho}_0 / \partial z^\alpha$  are not identical, however, and it is important to use the proper first-order gyroradius in order to obtain the correct first-order polarization and magnetization.

In particle phase space, the guiding-center position  $\mathbf{X}$  is expressed in terms of the guiding-center transformation (29) as  $\mathbf{X} = \mathbf{x} - \epsilon \boldsymbol{\rho}_0 - \epsilon^2 \boldsymbol{\rho}_1 + \cdots \equiv \mathbb{T}_{\text{gc}} \mathbf{x}$ , where the first-order displacement  $G_1^\mathbf{x} \equiv -\boldsymbol{\rho}_0$  defines the lowest-order gyroradius, while the second-order correction is

$$\begin{aligned} \boldsymbol{\rho}_1 &= -G_2^\mathbf{x} - \frac{1}{2} \left[ \boldsymbol{\rho}_0 \cdot \nabla \boldsymbol{\rho}_0 - \left( G_1^\mu \frac{\partial \boldsymbol{\rho}_0}{\partial \mu} + G_1^\theta \frac{\partial \boldsymbol{\rho}_0}{\partial \theta} \right) \right] \\ &\equiv \left( \frac{1}{2} \boldsymbol{\rho}_0 \cdot \nabla \ln B - \rho_{\parallel} \tau \right) \boldsymbol{\rho}_0 + \left( \frac{1}{2} \boldsymbol{\rho}_0 \cdot \nabla \hat{\mathbf{b}} \cdot \boldsymbol{\rho}_0 - G_{2\parallel} \right) \hat{\mathbf{b}}. \end{aligned} \quad (47)$$

The gyroangle-averaged particle gyroradius (47) is

$$\langle \boldsymbol{\rho}_1 \rangle = \frac{\mu B}{2m\Omega^2} \left[ (\nabla \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}} + \nabla_{\perp} \ln B \right] \equiv -\nabla \cdot \left( \left\langle \frac{\boldsymbol{\rho}_0 \boldsymbol{\rho}_0}{2} \right\rangle \right) - \frac{\mu B (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}})}{2m\Omega^2}, \quad (48)$$

where we used  $\langle G_{2\parallel} \rangle \equiv 0$  from Eq. (34).

In guiding-center phase space, on the other hand, where the particle position  $\mathbf{x}$  is expressed in terms of the inverse guiding-center transformation as  $\mathbf{x} = \mathbf{X} + \boldsymbol{\rho}_{0\text{gc}} + \boldsymbol{\rho}_{1\text{gc}} + \cdots \equiv \mathbb{T}_{\text{gc}}^{-1} \mathbf{X}$ , the first-order correction is

$$\begin{aligned} \boldsymbol{\rho}_{1\text{gc}} &= -G_2^\mathbf{x} + \frac{1}{2} \left[ \boldsymbol{\rho}_0 \cdot \nabla \boldsymbol{\rho}_0 - \left( G_1^\mu \frac{\partial \boldsymbol{\rho}_0}{\partial \mu} + G_1^\theta \frac{\partial \boldsymbol{\rho}_0}{\partial \theta} \right) \right] \\ &\equiv - \left( G_1^\theta + \boldsymbol{\rho}_0 \cdot \mathbf{R} \right) \frac{\partial \boldsymbol{\rho}_0}{\partial \theta} - \left( G_1^\mu + 2\mu \rho_{\parallel} \tau \right) \frac{\partial \boldsymbol{\rho}_0}{\partial \mu} - \left( G_{2\parallel} + \frac{1}{2} \boldsymbol{\rho}_0 \cdot \nabla \hat{\mathbf{b}} \cdot \boldsymbol{\rho}_0 \right) \hat{\mathbf{b}} \end{aligned} \quad (49)$$

where the same generating vector fields  $(G_1, G_2, \cdots)$  are used and we inserted the identity

$$\boldsymbol{\rho}_0 \cdot \nabla \boldsymbol{\rho}_0 \equiv -(\mu \boldsymbol{\rho}_0 \cdot \nabla \ln B) \frac{\partial \boldsymbol{\rho}_0}{\partial \mu} - (\boldsymbol{\rho}_0 \cdot \mathbf{R}) \frac{\partial \boldsymbol{\rho}_0}{\partial \theta} - (\boldsymbol{\rho}_0 \cdot \nabla \hat{\mathbf{b}} \cdot \boldsymbol{\rho}_0) \hat{\mathbf{b}}$$

in obtaining the last expression for  $\boldsymbol{\rho}_{1\text{gc}}$ .

Lastly, we note that the first-order particle and guiding-center gyroradii (47) and (49) are both gyro-gauge-invariant. Moreover, according to Eqs. (21) and (27), it is the guiding-center first-order gyroradius vector (49) that must be used and, hence, we will use  $\langle \boldsymbol{\rho}_{1\text{gc}} \rangle$  to compute the first-order guiding-center polarization (21) and  $\tilde{\boldsymbol{\rho}}_{1\text{gc}} \equiv \boldsymbol{\rho}_{1\text{gc}} - \langle \boldsymbol{\rho}_{1\text{gc}} \rangle$  to compute the first-order guiding-center magnetization (27).

### C. First-order Guiding-center Polarization

The gyroangle-averaged guiding-center gyroradius (49) is

$$\langle \boldsymbol{\rho}_{1\text{gc}} \rangle \equiv \frac{\hat{\mathbf{b}}}{\Omega} \times \frac{d_{\text{gc}} \mathbf{X}}{dt} + \nabla \cdot \left( \left\langle \frac{\boldsymbol{\rho}_0 \boldsymbol{\rho}_0}{2} \right\rangle \right) + \frac{\mu B (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}})}{2m\Omega^2}, \quad (50)$$

where we used Eq. (48). When we combine these results into the first-order guiding-center polarization (21), we obtain

$$\mathbf{P}_{1gc} = \frac{\hat{\mathbf{b}}}{\Omega} \times \left[ \int \bar{F} \left( e \frac{d_{gc} \bar{\mathbf{X}}}{dt} \right) d^3 \bar{v} + c \nabla \times \left( \frac{1}{2} \int \left( -\bar{\mu} \hat{\mathbf{b}} \right) \bar{F} d^3 \bar{v} \right) \right]. \quad (51)$$

This expression combines the classical first-order (dipole) guiding-center polarization obtained previously [9] as well as the first-order quadrupole guiding-center polarization.

#### D. First-order Guiding-center Magnetization

The calculation of the first-order guiding-center magnetization (27) requires the guiding-center push-forward of the particle velocity  $\mathbf{v} = d\mathbf{x}/dt$ , which is defined as

$$\mathbf{V}_{gc} \equiv \mathbf{T}_{gc}^{-1} \mathbf{v} \equiv \frac{d_{gc} \mathbf{X}}{dt} + \frac{d_{gc} \boldsymbol{\rho}_{gc}}{dt} = \mathbf{V}_{0gc} + \epsilon \mathbf{V}_{1gc} + \dots, \quad (52)$$

where

$$\mathbf{V}_{0gc} \equiv \frac{d_{gc}^{(0)} \mathbf{X}}{dt} + \frac{d_{gc}^{(0)} \boldsymbol{\rho}_{0gc}}{dt} = v_{\parallel} \hat{\mathbf{b}} + \Omega \frac{\partial \boldsymbol{\rho}_{0gc}}{\partial \theta}$$

denotes the lowest-order guiding-center (particle) velocity and its first-order correction is

$$\mathbf{V}_{1gc} \equiv \langle \mathbf{V}_{1gc} \rangle + \tilde{\mathbf{V}}_{1gc}, \quad (53)$$

where the gyroangle-independent part is

$$\langle \mathbf{V}_{1gc} \rangle \equiv \frac{d_{gc}^{(1)} \mathbf{X}}{dt} = \left( \hat{\mathbf{b}} \times \frac{d_{gc} \mathbf{X}}{dt} \right) \times \hat{\mathbf{b}} \quad (54)$$

and the gyroangle-dependent part is

$$\tilde{\mathbf{V}}_{1gc} \equiv \frac{d_{gc}^{(0)} \tilde{\boldsymbol{\rho}}_{1gc}}{dt} + \frac{d_{gc}^{(1)} \boldsymbol{\rho}_{0gc}}{dt} = \Omega \left( \frac{\partial \tilde{\boldsymbol{\rho}}_{1gc}}{\partial \theta} + \rho_{\parallel} \hat{\mathbf{b}} \cdot \nabla^* \boldsymbol{\rho}_{0gc} \right). \quad (55)$$

Here, we easily show that the guiding-center velocity  $\mathbf{V}_{gc}$  satisfies the following identities:  $\langle \mathbf{V}_{gc} \rangle \equiv d_{gc} \mathbf{X}/dt$  and  $H_{gc} = (m/2) |\mathbf{V}_{gc}|^2$ .

Lastly, by inserting Eqs. (49)-(50) and (55) into Eq. (27), we obtain the first-order guiding-center magnetization

$$\mathbf{M}_{1gc} = \int \bar{\rho}_{\parallel} \bar{F} \left[ \bar{\mu} \left( \tau \hat{\mathbf{b}} + \frac{3}{2} B \nabla \times (B^{-1} \hat{\mathbf{b}}) \right) + \frac{m \bar{v}_{\parallel}^2}{B} \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \right] d^3 \bar{v}. \quad (56)$$

We note, however, that the first-order guiding-center magnetization (56) vanishes if the gyrocenter Vlasov distribution  $\bar{F}$  is a Maxwellian in  $\bar{v}_{\parallel}$ . In most applications of gyrokinetic theory, the zeroth-order guiding-center magnetization (26) is therefore sufficient.

#### IV. SUMMARY

We have presented the derivation of guiding-center polarization and magnetization effects as a simple extension of the standard form of modern gyrokinetic theory by keeping higher-order corrections to the guiding-center gyroradius (49) in the first-order gyrocenter Hamiltonian (15) [e.g., Eq. (17)].

From the variational derivations of the guiding-center polarization and magnetization, we have recovered the classical first-order guiding-center polarization (51) and the classical zeroth-order guiding-center magnetization (26). These results confirm that the two-step perturbation analysis used in modern gyrokinetic theory yield a consistent set of gyrokinetic Vlasov-Maxwell equations that include all first-order polarization and magnetization effects.



### Acknowledgments

This work was supported by a U. S. Dept. of Energy grant under contract No. DE-FG02-09ER55005.

### Appendix A: Reduced Polarization and Magnetization

In this Appendix, we present a brief summary of the derivation of reduced polarization and magnetization effects [16] induced by a general near-identity phase-space transformation  $\mathbf{z} \rightarrow \bar{\mathbf{z}} \equiv \mathcal{T}_\epsilon \mathbf{z}$ . The dynamical reduction introduced by this phase-space transformation yields the reduced Vlasov equation

$$\frac{\partial \bar{f}}{\partial t} + \frac{1}{\bar{f}} \frac{\partial}{\partial \bar{z}^\alpha} \left( \bar{f} \frac{d_\epsilon \bar{z}^\alpha}{dt} \bar{f} \right) = 0, \quad (\text{A1})$$

where  $\bar{f}$  denotes the Jacobian of the transformation  $\mathcal{T}_\epsilon$  and the reduced Hamilton equations  $d_\epsilon \bar{z}^\alpha / dt \equiv \{\bar{z}^\alpha, \bar{H}\}_\epsilon$  are represented in terms of a reduced Hamiltonian  $\bar{H}$  and a reduced Poisson bracket  $\{, \}_\epsilon$  (not necessarily canonical).

Reduced polarization and magnetization effects are associated with the reduced displacement  $\bar{\boldsymbol{\rho}}_\epsilon \equiv \mathcal{T}_\epsilon^{-1} \mathbf{x} - \bar{\mathbf{x}}$ . We begin with the push-forward derivation of the reduced polarization generated by the phase-space transformation  $\mathcal{T}_\epsilon$ . The particle charge density (summation over particle species is implied)

$$\varrho \equiv e \int \bar{f} \delta^3(\bar{\mathbf{x}} + \bar{\boldsymbol{\rho}}_\epsilon - \mathbf{r}) d^6 \bar{\mathbf{z}} \equiv \bar{\varrho} - \nabla \cdot \bar{\mathbf{P}} \quad (\text{A2})$$

is expressed in terms of the reduced charge density

$$\bar{\varrho} \equiv e \int \bar{f} \delta^3(\bar{\mathbf{x}} - \mathbf{r}) d^6 \bar{\mathbf{z}} = e \int \bar{f} d^3 \bar{\mathbf{v}}, \quad (\text{A3})$$

and the reduced polarization charge density  $\bar{\varrho}_{\text{pol}} \equiv -\nabla \cdot \bar{\mathbf{P}}$ , where the reduced polarization is defined as a multipole expansion (dipole + quadrupole + ...) associated with increasing powers of  $\bar{\boldsymbol{\rho}}_\epsilon$ :

$$\bar{\mathbf{P}} \equiv e \int \bar{f} \bar{\boldsymbol{\rho}}_\epsilon d^3 \bar{\mathbf{v}} - \nabla \cdot \left( \frac{e}{2} \int \bar{f} \bar{\boldsymbol{\rho}}_\epsilon \bar{\boldsymbol{\rho}}_\epsilon d^3 \bar{\mathbf{v}} \right) + \dots \quad (\text{A4})$$

Next, we consider the push-forward derivation of the reduced magnetization generated by the phase-space transformation  $\mathcal{T}_\epsilon$ . This derivation uses the push-forward transformation of the particle velocity  $\mathbf{v} \equiv d\mathbf{x}/dt$ :

$$\bar{\mathbf{v}} = \mathcal{T}_\epsilon^{-1} \mathbf{v} \equiv \left[ \mathcal{T}_\epsilon^{-1} \left( \frac{d}{dt} \mathcal{T}_\epsilon \right) \right] \mathcal{T}_\epsilon^{-1} \mathbf{x} = \frac{d_\epsilon \bar{\mathbf{x}}}{dt} + \frac{d_\epsilon \bar{\boldsymbol{\rho}}_\epsilon}{dt}. \quad (\text{A5})$$

The particle current density

$$\mathbf{J} \equiv e \int \bar{f} \left( \frac{d_\epsilon \bar{\mathbf{x}}}{dt} + \frac{d_\epsilon \bar{\boldsymbol{\rho}}_\epsilon}{dt} \right) \delta^3(\bar{\mathbf{x}} + \bar{\boldsymbol{\rho}}_\epsilon - \mathbf{r}) d^6 \bar{\mathbf{z}} = \bar{\mathbf{J}} + \frac{\partial \bar{\mathbf{P}}}{\partial t} + c \nabla \times \bar{\mathbf{M}}, \quad (\text{A6})$$

is thus expressed in terms of the reduced current density

$$\bar{\mathbf{J}} \equiv e \int \bar{f} \frac{d_\epsilon \bar{\mathbf{x}}}{dt} d^3 \bar{\mathbf{v}}, \quad (\text{A7})$$

where  $\bar{\mathbf{J}}_{\text{pol}} \equiv \partial \bar{\mathbf{P}} / \partial t$  is the reduced polarization current density, and the reduced magnetization current density  $\bar{\mathbf{J}}_{\text{mag}} \equiv c \nabla \times \bar{\mathbf{M}}$ . If we expand Eq. (A6) in powers of  $\bar{\boldsymbol{\rho}}_\epsilon$ , we obtain

$$\mathbf{J} = e \int \bar{f} \left( \frac{d_\epsilon \bar{\mathbf{x}}}{dt} + \frac{d_\epsilon \bar{\boldsymbol{\rho}}_\epsilon}{dt} \right) d^3 \bar{\mathbf{v}} - \nabla \cdot \left[ e \int \bar{f} \bar{\boldsymbol{\rho}}_\epsilon \left( \frac{d_\epsilon \bar{\mathbf{x}}}{dt} + \frac{d_\epsilon \bar{\boldsymbol{\rho}}_\epsilon}{dt} \right) d^3 \bar{\mathbf{v}} \right] + \dots \quad (\text{A8})$$

Next, we take the partial time derivative of the reduced polarization (A4), with the reduced Vlasov (A1), we can insert

$$e \int \bar{f} \left( \frac{d_\epsilon \bar{\boldsymbol{\rho}}_\epsilon}{dt} \right) d^3 \bar{\mathbf{v}} = \frac{\partial \bar{\mathbf{P}}}{\partial t} + \nabla \cdot \left\{ e \int \bar{f} \left[ \frac{d_\epsilon \bar{\mathbf{x}}}{dt} \bar{\boldsymbol{\rho}}_\epsilon + \frac{1}{2} \left( \bar{\boldsymbol{\rho}}_\epsilon \frac{d_\epsilon \bar{\boldsymbol{\rho}}_\epsilon}{dt} + \frac{d_\epsilon \bar{\boldsymbol{\rho}}_\epsilon}{dt} \bar{\boldsymbol{\rho}}_\epsilon \right) \right] \right\} + \dots$$

into Eq. (A8) and obtain

$$\mathbf{J} = \bar{\mathbf{J}} + \frac{\partial \bar{\mathbf{P}}}{\partial t} + \nabla \cdot \left\{ e \int \bar{f} \left[ \left( \frac{d_\epsilon \bar{\mathbf{x}}}{dt} \bar{\boldsymbol{\rho}}_\epsilon - \bar{\boldsymbol{\rho}}_\epsilon \frac{d_\epsilon \bar{\mathbf{x}}}{dt} \right) + \frac{1}{2} \left( \frac{d_\epsilon \bar{\boldsymbol{\rho}}_\epsilon}{dt} \bar{\boldsymbol{\rho}}_\epsilon - \bar{\boldsymbol{\rho}}_\epsilon \frac{d_\epsilon \bar{\boldsymbol{\rho}}_\epsilon}{dt} \right) \right] \right\} + \dots \quad (\text{A9})$$

Lastly, by using the identity  $\nabla \cdot (\mathbf{B}\mathbf{A} - \mathbf{A}\mathbf{B}) \equiv \nabla \times (\mathbf{A} \times \mathbf{B})$ , for two arbitrary vector fields  $\mathbf{A}$  and  $\mathbf{B}$ , we obtain an expression for the reduced magnetization

$$\bar{\mathbf{M}} \equiv \frac{e}{c} \int \bar{f} \left[ \bar{\boldsymbol{\rho}}_\epsilon \times \left( \frac{1}{2} \frac{d_\epsilon \bar{\boldsymbol{\rho}}_\epsilon}{dt} + \frac{d_\epsilon \bar{\mathbf{x}}}{dt} \right) \right] d^3\bar{v} + \dots \quad (\text{A10})$$

In Eq. (A10), the term associated with  $(1/2)\bar{\boldsymbol{\rho}}_\epsilon \times d_\epsilon \bar{\boldsymbol{\rho}}_\epsilon/dt$  represents the intrinsic reduced magnetization while the term  $\bar{\boldsymbol{\rho}}_\epsilon \times d_\epsilon \bar{\mathbf{x}}/dt$  represents the moving-electric-dipole contribution.

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